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AN UPPER BOUND FOR THE LEAST DILATATION

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ABSTRACT. We given an upper bound for the least dilatation arising from a pseudo-Anosov map of a closed surface of genus greater or equal to three.

1. INTRODUCTION AND BACKGROUND

Introduction. Throughout the paper, $F = F_g$ will be a closed surface of genus g with negative Euler characteristic. Suppose that $(\mathcal{F}, \nu)$ is a measured foliation (see [FLP]) and ϕ is an orientation preserving homeomorphism of F, then we define $\phi(\mathcal{F})$ to be the foliation whose leaves are the images of the leaves of $\mathcal{F}$. Furthermore, $\phi_*(\nu)$ is a measure on $\phi(\mathcal{F})$ that is defined as the push forward of the measure ν under ϕ. To be more explicit, if α is an arc transverse to the foliation $\phi(\mathcal{F})$, then $\phi_*(\nu)(\alpha) = \nu(\phi^{-1}(\alpha))$. We define $\bar{\phi}(\mathcal{F}, \nu) = (\phi(\mathcal{F}), \phi_*(\nu))$.

An orientation preserving homeomorphism ϕ of F is *pseudo-Anosov* (or *p.A.*) if there is a pair of transverse arational (i.e. no closed leaves) measured foliations $(\mathcal{F}, \nu)$ and $(\mathcal{F}^\perp, \nu^\perp)$ in F, such that $\bar{\phi}(\mathcal{F}, \nu) = (\mathcal{F}, \lambda\nu)$ and $\bar{\phi}(\mathcal{F}^\perp, \nu^\perp) = (\mathcal{F}^\perp, (1/\lambda)\nu^\perp)$, for some $\lambda > 1$. λ is called the *dilatation* of ϕ, and we define the 'spectrum' of F as

$$\operatorname{Spec}(F) = \{\log\lambda : \lambda \text{ is the dilatation of a p.A. self-map of } F\} \subset \mathbb{R}.$$

$\operatorname{Spec}(F)$ has a geometric interpretation as the collection of Teichmüller distances between Riemann surfaces of the same topological type as F (see [Ab]). Furthermore, a pseudo-Anosov map ϕ realizes the smallest topological entropy in its homotopy class and the topological entropy of ϕ is given by the logarithm of its dilatation (see [FLP]). It is known that $\operatorname{Spec}(F)$ is discrete (see [AY]).

The main goal of this note is to show that the smallest element δ_g of $\operatorname{Spec}(F_g)$ allows the upper bound

$$\delta_g \leq \frac{\log 6}{g}, \quad \text{for all } g \geq 3.$$

This is an improvement of the upper bound given in [P2].

Background. We begin by establishing some notation. The group of $l \times l$ matrices over $\mathbb{Z}$ is denoted by $\operatorname{Mat}_l(\mathbb{Z})$, and the *spectrum* of $A \in \operatorname{Mat}_l(\mathbb{Z})$ is the set of eigenvalues of A listed with multiplicity. We say that an eigenvalue λ in

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the spectrum of A has *maximum modulus* if the modulus of λ strictly exceeds the modulus of any other element of the spectrum of A. Suppose next that A is an $n \times m$ matrix with $n, m \geq 1$, then by an expression of the form $A > 0$, $A \geq 0$, etc. we mean that the relevant relation holds for each component of A. A matrix $A \in \mathrm{Mat}_l(\mathbb{Z})$ is said to be *Perron-Frobenius* or *P.F.* if $A \geq 0$ and for some $n \geq 1$ we have $A^n > 0$.

Perron-Frobenius matrices have the following well known spectral properties (see [Ga]).

Theorem (Perron-Frobenius). *The spectrum of a P.F. matrix A contains an element λ of maximum modulus that is positive real with corresponding eigenvector x^* strictly positive. x^* is the unique positive eigenvector and λ is a simple root of the characteristic polynomial of A.*

The main tool in the present investigation is the theory of train tracks as introduced by Thurston [Th]. The material that is needed for the present investigation is collected in the section on train tracks in [Ba]. We refer the reader who wishes more information on surface homeomorphisms and train tracks to [CB, FLP, Th, Pa, PP] and the monograph on train tracks [HP]. We remark in particular that if τ is a train track embedded in the surface F, and if $V(\tau)$ denotes the set of measures on τ, then there is an embedding $\mathscr{I}$ of $V(\tau)$ into the space of (equivalence classes of) measured foliations $\mathscr{MF}(F)$. Moreover, if τ is invariant under a homeomorphism ϕ of F, then there are induced maps $\hat{\phi}\colon V(\tau) \to V(\tau)$ and $\bar{\phi}\colon \mathscr{MF}(F) \to \mathscr{MF}(F)$ such that $\mathscr{I} \circ \hat{\phi} = \bar{\phi} \circ \mathscr{I}$.

2. An upper bound by example

A class of pseudo-Anosov maps. We describe in this section the map that yields the upper bound mentioned in the introduction.

If k is a simple closed curve embedded in F, then we denote the (right handed) Dehn twist along k by τ_k. (For a definition of Dehn twist see for example [Ba].) For $g \geq 3$, we take the surface F_g as a sphere with g handles attached so that F_g is invariant under the rotation $\rho = \rho_g$ by $2\pi/g$ about the axis L through the north and south pole of the sphere. Figure 1 shows the case $g = 3$. We then define

$$\psi = \psi_g = \rho \circ \tau_c^{-1} \circ \tau_b \circ \tau_a .$$

Whenever the dependency of an object on the genus g is clear we will simplify notation by suppressing this dependency. For example the maps ρ, τ_c^{-1}, τ_b and τ_a depend on g.

Our first goal is to show that ψ_g is (isotopic to) a pseudo-Anosov homeomorphism using the following pseudo-Anosov recognition theorem (see [CB]).

Theorem (Casson). *A homeomorphism ϕ of F is isotopic to a p.A. map if there is a ϕ invariant train track τ that fills F, such that no proper subtrack τ' of τ is invariant under ϕ, and such that if τ itself is a subtrack of a ϕ invariant train track τ'' (not necessarily proper) then the induced map $\phi''\colon V(\tau'') \to V(\tau'')$ has no nonzero fixed point.*

The train track that will allow us to apply the previous theorem is $\tau(g)$ as shown in Figure 2 in case $g = 3$. It is clear how $\tau(g)$ is defined (up to isotopy) for $g \geq 3$. We isotope $\tau(g)$ so that $\tau(g)$ is invariant under ρ. Note that

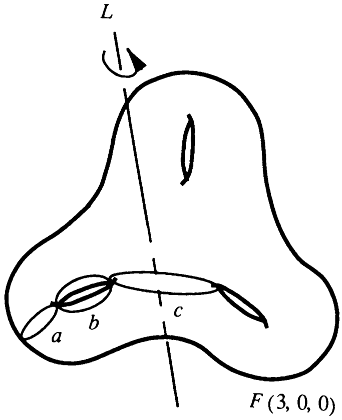

FIGURE 1. The surface F_g, in case $g = 3$

$\tau(g)$ has as complementary regions $2g$ tri-gons and 2 g-gons, hence $\tau(g)$ fills the surface. The number of branches of $\tau(g)$ is $7g$.

We need to be more precise as to how ψ_g acts on F_g and to that end we use the notation for branches of $\tau(g)$ as illustrated in Figure 2 in case $g = 3$. Clearly we can isotope $\tau_b \circ \tau_a$ so that it fixes the switches of $\tau(g)$ and all of the branches except for b_1, b_2, and b_3. We also isotope τ_c^{-1} such that it fixes the switches of $\tau(g)$ and only moves the branches b_4 and b_5.

By choosing the central ties of a standard tie neighborhood of $\tau(g)$ carefully, we may assume that $\tau_b \circ \tau_a(b_1)$ intersects the central ties corresponding to the branches of $\{\rho^{g-1}(b_5), \rho^{g-1}(a_2), b_3, a_1, b_4\}$; also $\tau_b \circ \tau_a(b_2)$ intersects the central tie corresponding to the branch b_1, and $\tau_b \circ \tau_a(b_3)$ intersects the central ties corresponding to the branches of $\{\rho^{g-1}(b_4), b_2, b_5\}$. Similarly, we may assume that $\tau_c^{-1}(b_4)$ intersects the central ties corresponding to the branches of $\{\rho(b_3), a_2, b_3, a_1, b_4\}$, and $\tau_c^{-1}(b_5)$ intersects the ones corresponding to $\{b_3, a_1, \rho(b_3), a_2, b_5\}$.

For future reference, we summarize

Lemma 1. *For $g \geq 3$, $\tau(g)$ is a ψ_g invariant train track that fills F_g.*

Note that if we define $\overline{B} = \bigcup_{i=0}^{g-1} B_i$, where $B_i = \{\rho^i(b_1), \dots, \rho^i(b_5)\}$, for $0 \leq i \leq g-1$, then $\tau_c^{-1} \circ \tau_b \circ \tau_a$ only affects the branches B_0. Whenever we need to assume that the branches of $\tau(g)$ are ordered, we choose the following ordering

$$(\rho^0(b_1), \dots, \rho^0(b_5), \dots, \rho^{g-1}(b_1), \dots, \rho^{g-1}(b_5),$$
$$\rho^0(a_1), \rho^0(a_2), \dots, \rho^{g-1}(a_1), \rho^{g-1}(a_2)).$$

Note that this defines an induced ordering of the branches $\overline{B}$.

If ϕ is a composition of ρ, τ_c^{-1}, and $\tau_b \circ \tau_a$, then we define $\mathrm{Inc}(\phi)$ to be the incidence matrix of ϕ with respect to the ordering of the branches of τ specified above. We remark that $\mathrm{Inc}(\phi)$ is well defined by our convention of how ρ, τ_c^{-1}, and $\tau_b \circ \tau_a$ act on $\tau(g)$. Note that the distributive property of the 'hat' operator (i.e. $\widehat{\phi_1\phi_2} = \hat{\phi}_1\hat{\phi}_2$) implies that for example $\mathrm{Inc}(\tau_c^{-1} \circ \tau_b \circ \tau_a) = \mathrm{Inc}(\tau_c^{-1})\,\mathrm{Inc}(\tau_b \circ \tau_a)$.

The following lemma is crucial in showing that ψ_g is p.A.

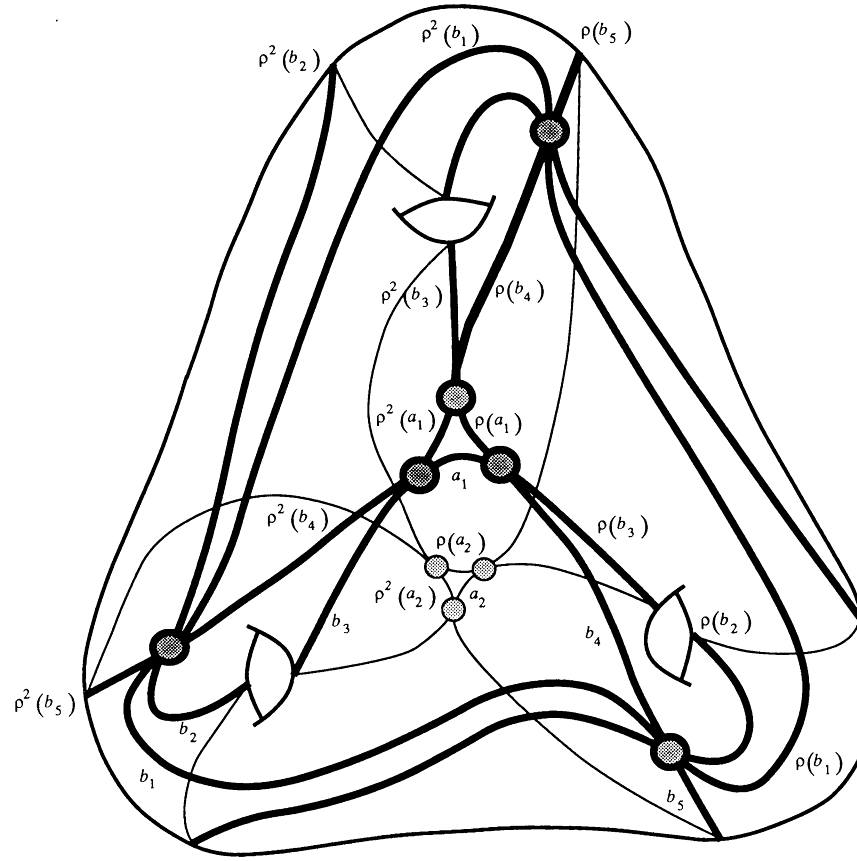

FIGURE 2. Invariant train track for ψ_g, in case $g = 3$

Lemma 2. *If $w^{(0)} \in \mathbb{R}^{7g}$ represents a nonzero weight on $\tau(g)$ (we do not require that the switch conditions hold), such that $w^{(0)}$ is positive on at least one branch of $\overline{B}$, then there is an n_0 such that for $n > n_0$, $\mathrm{Inc}(\psi^n)w^{(0)} > 0$.*

Proof. Recall that $\tau_c^{-1} \circ \tau_b \circ \tau_a$ only moves the branches B_0. This will be most important and used repeatedly without being mentioned.

Claim 1. Suppose that $T \in \mathrm{Mat}_5(\mathbb{Z})$ consists of those rows and columns of the incidence matrix $\mathrm{Inc}(\tau_c^{-1} \circ \tau_b \circ \tau_a)$ that correspond to the branches B_0, then $T^4 > 0$.

We readily see that

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

and compute.

Claim 2. If $w \in \mathbb{R}^{7g}$ represents a weight on $\tau(g)$ that is positive on a set $B_0' \subset B_0$, then the weight $\mathrm{Inc}(\psi^{g-1} \circ \rho)w$ is also positive on B_0'.

This is immediate.

Claim 3. If for some $0 \leq t \leq g-1$, $w \in \mathbb{R}^{7g}$ is a weight that is positive on *all* branches of B_t, then $\mathrm{Inc}(\psi^g)w$ is also positive on the branches B_t.

Indeed, $w' = \mathrm{Inc}(\psi^{g-t})w$ is positive on the branches B_0, and as each row of T contains at least one positive entry, $w'' = \mathrm{Inc}(\tau_c^{-1} \circ \tau_b \circ \tau_a)w'$ is still positive on the branches B_0. But then $\mathrm{Inc}(\psi^{t-1}\rho)w''$ is positive on B_t as desired.

Claim 4. If w is a weight that is positive on the branches of B_0, then $w^* = M(\tau_c^{-1} \circ \tau_b \circ \tau_a)w$ is positive on $B_0 \cup \{a_1, a_2\}$.

Indeed, we already remarked in the proof of Claim 3 that w^* is positive on branches of B_0, and the fact that w^* is positive on the branches a_1 and a_2 can be seen directly from our convention of how τ_c^{-1} and $\tau_b \circ \tau_a$ acts on $\tau(g)$.

We now take $w^{(0)}$ as in the statement of the lemma, hence there exists $t \in \{0, \dots, g-1\}$, such that $w^{(0)}$ is positive on a branch of B_t.

We first prove that $\mathrm{Inc}(\psi^n)w^{(0)}$, for n large enough, is positive on the branches of $\overline{B}$. The proof is by induction on the number of i such that $\mathrm{Inc}(\psi^n)w^{(0)}$, for n large enough, is positive on the branches of B_i. We take the subscripts of B_i as elements of the cyclic group on g elements.

For the basis step, we remark that $w^{(1)} = \mathrm{Inc}(\psi^{g-t})w^{(0)}$ is positive on a branch of B_0. It follows from Claim 1 and Claim 2 that $w^{(2)} = \mathrm{Inc}(\psi^{4g})w^{(1)}$ is positive on all branches of B_0.

For the induction step we assume that for some n, $w^{(3)} = \mathrm{Inc}(\psi^n)w^{(0)}$ is positive on the branches of $B_0, \dots, B_j$, for some $j \in \{0, \dots, g-1\}$. We first note that $w^{(4)} = \mathrm{Inc}(\psi^{g-j})w^{(3)}$ is positive on the branches of B_0. But then $w^{(5)} = \mathrm{Inc}(\tau_c^{-1} \circ \tau_b \circ \tau_a)w^{(4)}$ is positive on the branches $B_0 \cup \{\rho(b_3)\}$ (as follows from our convention of how ψ_g acts on $\tau(g)$), hence $w^{(6)} = \mathrm{Inc}(\psi^{g-2}\rho)w^{(5)}$ is positive on the branches $B_{g-1} \cup \{b_3\}$. Using the fact that $w^{(6)}$ is positive on a branch of B_0, we see as above, using Claim 1 and Claim 2, that $w^{(7)} = \mathrm{Inc}(\psi^{4g})w^{(6)}$ is positive on all branches of B_0. We conclude that $w^{(8)} = \mathrm{Inc}(\psi^{j+1})w^{(7)}$ is positive on the branches of B_{j+1}. To complete the induction step, note first that $w^{(8)} = \mathrm{Inc}(\psi^{6g})w^{(3)}$. As $w^{(3)}$ is positive on the branches of $B_0, \dots, B_j$, so is $w^{(8)}$ as follows from Claim 3.

We showed that for some $m \geq 1$ we have that $\mathrm{Inc}(\psi^m)w^{(0)}$ is positive on the branches of $\overline{B}$. To finish the proof of the lemma, we use Claim 4 to see that $\mathrm{Inc}(\psi^{m+g})w^{(0)}$ is positive on all branches of τ and the claim follows by taking $n_0 = m + g$. Q.E.D.

In order to show that ψ_g is pseudo-Anosov, we need to analyze the incidence matrix $\mathrm{Inc}(\psi_g)$. We claim

Lemma 3. (a)

$$\mathrm{Inc}(\psi_g) = \begin{pmatrix} S_g & \overline{0} \\ C & P \end{pmatrix} \in \mathrm{Mat}_{7g}(\mathbb{Z}),$$

where $\overline{0}$ is the $5g \times 2g$ zero matrix and $P \in \mathrm{Mat}_{2g}(\mathbb{Z})$ is a permutation matrix. (Of course C is a $2g \times 5g$ matrix.) Moreover,

(b) *the 'small' incidence matrix $S = S_g \in \mathrm{Mat}_{5g}(\mathbb{Z})$ of ψ with respect to the branches $\overline{B}$ of $\tau(g)$ is P.F.*

Proof. For the proof of part (a) we remark that ρ and hence ψ induces a permutation of the branches $A = \{\rho^i(a_j) : 0 \leq i \leq g-1 \text{ and } j = 1, 2\}$. To be more specific, if w is a weight on $\tau(g)$ that is positive on a single branch

a of A, then $\mathrm{Inc}(\psi)w$ is positive only on $\rho(a)$. This explains the subblocks $\overline{0}$ and P of $\mathrm{Inc}(\psi)$.

For the proof of part (b) we define $e_i \in \mathbb{R}^{5g}$ to be the ith unit vector in $\mathbb{R}^{5g}$. To show that S is P.F. it is of course enough to show that there exists an $n \geq 1$ such that for $1 \leq i \leq 5g$ we have $S^n e_i > 0$. If for $1 \leq i \leq 5g$, we define $\bar{e}_i \in \mathbb{R}^{7g}$ to be the ith unit vector, then, using part (a), we readily see that it is enough to show that there exists an n such that for $1 \leq i \leq 5g$ we have that $\mathrm{Inc}(\psi^n)\bar{e}_i$ is positive on branches of $\overline{B}$. This, however, follows from Lemma 2. Q.E.D.

We can now prove

Proposition 4. *For $g \geq 3$, ψ_g is pseudo-Anosov.*

Proof. We need to check the conditions of Casson's theorem.

The first two conditions, namely that $\tau(g)$ fills F_g and is invariant under ψ_g follow from Lemma 1.

To show the third condition, suppose that $\tau' = \tau'(g)$ is a subtrack of $\tau = \tau(g)$ that is invariant under ψ_g. We will show that τ' cannot be a proper subtrack of $\tau(g)$.

We choose a nonzero measure μ' on τ' and extend μ' to a nonzero measure $\mu \in V(\tau)$ by defining $\mu(b) = \mu'(b)$, if b is a branch of τ' and $\mu(b) = 0$, otherwise. It is easy to see that any nonzero measure on τ is positive on some branch of $\overline{B}$, hence by Lemma 2, we can find $n > 0$ such that $\hat{\psi}^n(\mu)$ is positive on all branches of τ. ($\hat{\psi}$ being the self-map of $V(\tau)$ that is induced by ψ.) As we assumed that τ' is invariant under ψ, and as μ is zero on branches of $\tau \backslash \tau'$, the measure $\hat{\psi}^n(\mu)$ must also be zero on these branches. It follows that $\tau' = \tau$ as desired.

We are left to check the last condition in Casson's theorem, namely if τ is a subtrack of a ψ invariant train track $\tau'' = \tau''(g)$, then the induced map $\hat{\phi}'' \colon V(\tau'') \to V(\tau'')$ has no nonzero fixed point. An Euler characteristic argument shows that for τ'' to satisfy the fourth condition in the definition of train track, the branches of $\tau'' \backslash \tau$ must be contained in the closure of the two complementary g-gons of τ. But as ψ induces a rotation of these complementary g-gons, we readily see that τ'' can only be ψ invariant and satisfy the fourth condition in the definition of train track if $\tau'' = \tau$.

Assume now to derive a contradiction that $\mu \in V(\tau)$ is a nonzero fixed point of $\hat{\psi} \colon V(\tau) \to V(\tau)$, hence we have a nonzero fixed point $x \in \mathbb{R}^{7g}$ of $\mathrm{Inc}(\psi)$. As remarked above, any nonzero measure on τ is positive on some branch of $\overline{B}$, hence Lemma 2 applies to show that $x > 0$. It follows from Lemma 3 that x restricts to a nonzero fixed point $\bar{x}$ of S. But as $S \in \mathrm{Mat}_{5g}(\mathbb{Z})$ is P.F. and integral, $S^n \geq 1$, for n large enough, and as $5g > 1$, we see that S cannot have a nonzero fixed point. This contradiction shows that our assumption of $\hat{\psi}$ having a nonzero fixed point was absurd.

We showed that all the conditions in Casson's theorem are satisfied and conclude that ψ_g is indeed p.A. Q.E.D.

The upper bound. We will show

Proposition 5. *For $g \geq 3$, the dilatation λ_g of ψ_g satisfies*

$$\log \lambda_g \leq \frac{\log 6}{g}.$$

As a corollary we get our main result.

Theorem 6. *The smallest element* δ_g *in* $\operatorname{Spec}(F_g)$ *satisfies*

$$\delta_g \leq \frac{\log 6}{g}, \quad \textit{for } g \geq 3.$$

Before we prove the proposition some remarks are in order.

Remark. (a) One can define F_g and ϕ_g, for $g = 2$, but a computation shows that the dilatation λ_2 of ψ_2 satisfies $\lambda_2^2 \approx 6.018 > 6$.

(b) [P2] shows, using [P1], that $\rho \circ \tau_c \circ \tau_b^{-1} \circ \tau_a$ is p.A. with dilatation l_g that satisfies $l_g \leq (\log 11)/g$. The fact that ψ_g is p.A. does not follow from [P1] as we perform right-handed Dehn twists along two curves that intersect. The method in this note can be extended to produce a large class of p.A. maps. (See [Ba].)

Proof of Proposition 5.

Step 1. We choose $g \geq 3$ and show that we can find the dilatation of $\psi = \psi_g$ by spectrally analyzing the incidence matrix $\operatorname{Inc}(\psi)$. This is a standard technique. We then proceed to demonstrate that the dilatation of ψ is in fact given by the spectral radius of the smaller P.F. matrix $S = S_g$ (as in Lemma 3).

If we identify $V(\tau)$ with a closed cone in $\mathbb{R}^{7g}$, then $(V(\tau)\backslash\{0\})/\mathbb{R}_+$, where $\mathbb{R}_+$ denotes the positive real numbers, can be represented by a closed cell. $\operatorname{Inc}(\psi)$ induces a continuous self-map of this cell and we conclude from the Brouwer fixed point theorem that there exist $\sigma = \sigma_g > 0$ and $x \in \mathbb{R}^{7g}$ such that $\operatorname{Inc}(\psi)x = \sigma x$. Moreover, x corresponds to a nonzero measure on τ.

We show next that $\sigma > 1$ and that σ is the spectral radius of S. Indeed, we conclude from Lemma 3 that if we define $\bar{x} \in \mathbb{R}^{5g}$ to be the restriction of x onto its first $5g$ coordinates, then $S\bar{x} = \sigma\bar{x}$. Moreover, $x > 0$, as follows from Lemma 2, and hence $\bar{x} > 0$. The uniqueness statement of the Perron-Frobenius theorem shows that σ is the spectral radius of S. As S is P.F. and integral, we conclude that $\sigma > 1$.

We claim that σ equals the dilatation $\lambda = \lambda_g$ of ψ_g, and to that end we define $\mu \in V(\tau)$ to be the measure that corresponds to $x \in \mathbb{R}^{7g}$. We further define $(\mathscr{F}, \nu) = \mathscr{I}(\mu)$. We showed above that $\operatorname{Inc}(\psi)x = \sigma x$, hence $\hat{\psi}\mu = \sigma\mu$. As $\mathscr{I} \circ \hat{\psi} = \bar{\psi} \circ \mathscr{I}$, we conclude that $\bar{\psi}(\mathscr{F}, \nu) = (\mathscr{F}, \sigma\nu)$. But there is only one such foliation (class) for a p.A. map with $\sigma > 1$, hence $\sigma = \lambda$ as desired. The claim follows.

Step 2. We need to bound above the spectral radius λ of S. To that end we think of the matrix S and its powers as having g columns, the g elements of each column being 5×5 matrices. We find

$$S = \begin{pmatrix} A_1 & \bar{0} & \bar{0} & \bar{0} & \cdots & \bar{0} & \bar{0} & I \\ A_2 & \bar{0} & \bar{0} & \bar{0} & \cdots & \bar{0} & \bar{0} & \bar{0} \\ A_3 & I & \bar{0} & \bar{0} & \cdots & \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & I & \bar{0} & \cdots & \bar{0} & \bar{0} & \bar{0} \\ & & & & \vdots & & & \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \cdots & \bar{0} & I & \bar{0} \end{pmatrix} \in \operatorname{Mat}_{5g}(\mathbb{Z}),$$

where

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$I \in \mathrm{Mat}_5(\mathbb{Z})$ is the identity matrix, and $\overline{0} \in \mathrm{Mat}_5(\mathbb{Z})$ is the zero matrix.

Note that $A_1^2 = \overline{0}$. We define $A_4 = A_3 A_1 + A_2$ and $A_5 = A_1 A_3 + A_4$, and find that in case $g = 3$,

$$S^3 = \begin{pmatrix} A_5 & A_1 & A_3 \\ A_2 A_3 & A_2 & A_2 A_1 \\ A_3 + A_2 A_1 & A_3 & A_4 \end{pmatrix}.$$

Furthermore, for $g \geq 4$, we claim that

$$S^g = \left(\begin{array}{ccccccccccc} A_5 & A_1 & \overline{0} & \overline{0} & \overline{0} & \overline{0} & \cdots & \overline{0} & \overline{0} & \overline{0} & A_3 \\ A_2 A_3 & A_2 & A_2 A_1 & \overline{0} & \overline{0} & \overline{0} & \cdots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ A_3 & A_3 & A_4 & A_2 A_1 & \overline{0} & \overline{0} & \cdots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \hline \overline{0} & \overline{0} & A_3 & A_4 & A_2 A_1 & \overline{0} & \cdots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ & & & & & & \ddots & & & & \\ \overline{0} & \overline{0} & \overline{0} & \overline{0} & \overline{0} & \overline{0} & \cdots & \overline{0} & A_3 & A_4 & A_2 A_1 \\ A_2 A_1 & \overline{0} & \overline{0} & \overline{0} & \overline{0} & \overline{0} & \cdots & \overline{0} & \overline{0} & A_3 & A_4 \end{array} \right).$$

To show that, we choose $g \geq 4$, and define $c(i, n)$ as the ith column of S^n, for $1 \leq i \leq g$ and $1 \leq n \leq g$. We will only demonstrate that $c(i, g)$ is as asserted for $i \in \{3, \ldots, g-1\}$. The remaining cases are similar.

Note first that $c(i, 1)$ has I in its $(i-1)$th position and zero matrices $\overline{0}$ in the other positions. We further compute

$$\begin{aligned} c(i, g-i) &= (\overline{0}, \ldots, \overline{0}, I), \\ c(i, g-i+1) &= (I, \overline{0}, \ldots, \overline{0}), \\ c(i, g-i+2) &= (A_1, A_2, A_3, \overline{0}, \ldots, \overline{0}), \\ c(i, g-i+3) &= (\overline{0}, A_2 A_1, A_4, A_3, \overline{0}, \ldots, \overline{0}, \end{aligned}$$

and

$$c(i, g) = (\overline{0}, \ldots, \overline{0}, A_2 A_1, A_4, A_3, \overline{0}, \ldots, \overline{0}),$$

where in the last equation $A_2 A_1$ is in the $(i-1)$th position as desired.

We now choose $g \in \{3, 4, \ldots\}$, and note that as $\psi = \psi_g$ is p.A. with dilatation $\lambda = \lambda_g$, then ψ^g is p.A. with dilatation λ^g. We showed in Step 1 that λ is the spectral radius of the P.F. matrix S, hence λ^g is the spectral radius of the P.F. matrix S^g. The spectral radius λ^g of the P.F. matrix S^g satisfies (see [Ga])

$$\lambda^g = \min\left\{ \max_{1 \leq i \leq 5g} \frac{(S^g x)_i}{x_i} : x \in \mathbb{R}^{5g}, x > 0 \right\},$$

and hence

$$\lambda^g \leq \max_{1\leq i\leq 5g} \frac{(S^g x)_i}{x_i},$$

for any $x \in \mathbb{R}^{5g}$, $x > 0$.

If we let

$$y_1 = (.04, .22, 1.2, .82, .28), \quad y_2 = (.11, .65, 1.47, .48, .8),$$
$$y_3 = (.08, .47, 2.2, .48, .52), \quad \text{and} \quad x = (y_1, y_2, y_3, \ldots, y_3) \in \mathbb{R}^{5g},$$

then we compute that

$$\frac{(S^g x)_i}{x_i} \leq 6, \quad \text{for } i = 1, \ldots, 5g.$$

This completes the proof of the proposition. Q.E.D.

Remark. The estimate can be improved by better choice of x in the proof of Proposition 5. Some values for $(\lambda_g)^g$ (as provided by Matlab) and

g	$(\lambda_g)^g$
3	≈ 5.50
4	≈ 5.35
5	≈ 5.28
6	≈ 5.25
9	≈ 5.21

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